Nurturing Markets,
Widening Horizons

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Annexure 1

A detailed description of the Pienaar Choudhry method for extraction Zero rates and Par Yields from Traded Bond Prices

FIMMDA was entrusted with the task of developing a suitable model for the yield curve generation and streamlining the process for arriving at the prices for the G-Secs. Nelson Siegel Svensson and cubic spline zero curve were considered. A model based on Nelson Siegel Svensson provides a smooth zero curve; however it suffers from the demerit of a relatively higher price errors. This is because the model cannot incorporate multiple changes in curvature across various tenors. A cubic spline curve was considered to be appropriate for the Indian markets as the curve tracks the input price of various tenors and thereby produces a lower model error. In this approach the traded or proxy yields serve as the input for curve construction and a cubic spline is used to interpolate between the input yields to generate the curve. The cubic spline is a series of curves that is continuous at all the points. Each curve of the spline is of third order and has the form $Y = ax^3 + bx^2 + cx + d$ *where Y is zero-rate for the tenor 'x'.*

In the current method an optimization function is used to fit a natural cubic spline based zero curve to a set of traded bond prices. A yield curve is generated from the cubic spline based zero curve. The base research paper used as reference is authored by Rod Peinaar and Moorad Choudhry[1]. A simple cubic spline based implementation gives a good fit but leads to wavy forward curve. Hence a further smoothening constraint is applied to the optimization procedure that generates a curve which has minimum curvature and minimum price error. This smoothening leads to a better behavior of forward rates extracted from the zero rate curve.

The zero rate curve thus obtained is also used to extract par-yields for different maturities.

What is a cubic spline function?

A cubic spline function is a piecewise cubic polynomial function that passes through a given set of points in a smooth fashion. . The function takes the form $f = a_i + b_i\Delta + c_i\Delta^2 + d_i\Delta^3$; where 'i' represents the portion of the time axis where we want to measure zero-rate. If T_i represents the time to maturity of a traded bond, then between T_i and T_{i+1}, Δ takes values from 0 to $T_{i+1} - T_i$.

The time axis is divided in to regions by "knot points" at times T_i (usually the traded bond maturity in years). As we can see there is a different set of coefficients (a_i, b_i, c_i, d_i ...) describing the zero-rate curve between every T_i and T_{i+1}. The value of the cubic spline function as well as its first and second derivatives are the same when measured from either side of the knot point.

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For example, consider three consecutively maturing bonds, with maturity dates 14/06/2015, 17/08/2016 and 28/08/2017.
The current date is 29/07/2010; the time to maturity for each of these bonds is 4.875 years, 6.05 and 7.0805556 years respectively. So within the cubic spline framework the zero-rate between 4.875 and 6.05 years is described by one set of coefficients (a_i, b_i, c_i, d_i ...) and between 6.05 years and 7.08 years is described by another set of parameters ($a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$ ...). The value of Δ varies from 0 to (6.05-4.875=1.175) between 4.875 and 6.05 years. Similarly Δ takes the values from 0 to 1.0305 between 6.05 and 7.08 years. The zero-rate for 4.875 years is a_i, while that for 6.05 years is a_{i+1}.
Based on the constraints applicable for natural cubic spline (discussed in Annexure 1(a)) it is possible to describe the (b_i, c_i, d_i ...) coefficients in terms of the a_i coefficients and the maturities T_i . So in the optimization set-up the problem is to find values of a_i such that the squared difference between the model generated prices and traded prices is least

Pricing a coupon-bond given a zero rate curve

Suppose the values of the a_i coefficients for the cubic spline are as follows

Dates	a_i coefficients
29/07/2010	5.60%
02/07/2011	6.108%
03/09/2013	7.047%
14/06/2015	7.550%
17/08/2016	7.712%
28/08/2017	7.762%
03/05/2020	7.776%
15/02/2022	8.064%
02/07/2040	8.331%

Then as discussed earlier it is possible to obtain zero-rates (guesses) for any maturity. Consider the bond with coupon 9.39% and maturity date 02/07/2011. The (model) price for this bond is obtained using discount factors obtained from zero rates (see diagram below).

9.39 bond maturity i.e. 02/07/2011

Discount factors obtained by converting zero-rates

Coupon Dates	Coupons	DF (discount)	PV for coupon
02/01/2011	4.695	0.975503799	4.57999033
02/07/2011	104.695	0.945663949	99.0062871

Dirty Price	103.586277
Accrued interest	0.70425
Clean Price	102.882027

This clean price is the model price of the coupon bond.

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Pricing a T-bill given a zero-rate curve

The price of the T-bill is taken as 100 times the discount factor (from zero rates) for the maturity of the T-bill.

Fitting the zero-curve to reproduce traded bond prices

Let P represent the traded bond price (either T-bill or coupon bondand $\hat{P}$ model price (produced as above using a guess of the zero-curve). The values of coeficients (b_i, c_i, d_i ...) can be found from coefficent a_i. So by modifying a_i coefficient (using an optimization algorithmLevenberg Marquardt algorithm has been used in the current implementation) the cumulative price difference $\sum(\hat{P}-P)^2$ between the model price and traded price of the traded bonds is minimized.

Smoothening of zero-rate curve

To ensure that smooth forward rates are obtained from the zero-rate curve, a curvature term is added to the minimization of price differences. So instead of minimizing $\sum(\hat{P}-P)^2$ (bond price difference), $\sum(\hat{P}-P)^2+\int_{t}^{t}\lambda(t)*[f'']^2$ is minimized. The start point for this minimization is taken as the curve obtained by simply matching bond prices from the previous step.

f represents the zero-rate curve and f'' the second derivative of the curve.

Here λ (t) ('t' being time in years or maturity of the traded bond) is a function that augments the curvature, it is also called the VRP (variable roughness penalty) function.

Using this form of minimization leads to smoother curve but with possible mismatch in model price of bonds and traded price of bonds.

In the current implementation λ(t) is same as suggested by Daniel F. Waggoner[2]. It is a stepwise function and takes the following values:

$$\lambda(t)=0.1\_for\_t<1$$
$$\lambda(t)=100\_for\_1\le t<10$$
$$\lambda(t)=100000\_for\_10\le t$$

Results: Once the zero rate is obtained by the above method the par yield is derived from it. Par yield (or par rate) is the coupon rate for which the price of a coupon bond is equal to its par-value. For various maturities such as 0.25 years, 0.5 years..., let C represent the par-yield. Then one can solve the equation $C=\frac{1-df_{mat}}{\sum_k df_k}$, to obtain the par-yield for that maturity. Here df_{mat} represents the discount factor for the maturity date of the imaginary bond (say for 1.5 years maturity) and df_k represents the discount factors for the K-th coupon payment date of this imaginary coupon-bond. Again the discount factors are obtained from the zero-rate curve obtained from optimization earlier.

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Appendix 1(a)

Obtaining other coefficients from a_i coefficients

Matching the values of zero-rate, the first derivative of zero rate and the second derivative of the zero rate at the knot points T_i it is possible to write the following equations:

$$a_{i+1} = a_i + b_i\Delta + c_i\Delta^2 + d_i\Delta^3$$

$$b_{i+1} = b_i + \Delta_i(c_{i+1} - c_i) \quad \text{- (A)}$$

$$d_i = \frac{c_{i+1} - c_i}{3\Delta_i} \quad \text{- (B)}$$

For the coefficient c_i the following recurrence relation can be written:

$$\Delta_{i+1}c_{i+2} + \Delta_i c_i + 2(\Delta_i + \Delta_{i+1})c_{i+1} = -3\{(\frac{a_{i+1} - a_i}{\Delta_i}) - (\frac{a_{i+2} - a_{i+1}}{\Delta_{i+1}})\} \quad \text{- (C)}$$

Furthermore if there are in all “N” bonds, the values of c_1 and c_N are taken as zero (these conditions are called Natural cubic spline conditions), then the recurrence relation for the c_i coefficient s can be rewritten in the matrix form as a set of linear equations:

$$
\begin{array}{ll}
[2(\Delta_1 + \Delta_2)\ \ \Delta_2 \ldots\ldots \quad] & = [-3\{(\frac{a_2 - a_1}{\Delta_1}) - (\frac{a_3 - a_2}{\Delta_2})\}] \\
[\quad \Delta_2\ 2(\Delta_2 + \Delta_2)\ \Delta_3 \ldots \quad] & = [-3\{(\frac{a_3 - a_2}{\Delta_2}) - (\frac{a_4 - a_3}{\Delta_3})\}] \\
[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad] & = [\ldots\ldots\ldots] \\
[\ldots\ldots\ldots\ldots\ldots\ldots\Delta_{N-2}\ 2(\Delta_{N-2} + \Delta_{N-1})] & = [-3\{(\frac{a_{N-1} - a_{N-2}}{\Delta_{N-2}}) - (\frac{a_N - a_{N-1}}{\Delta_{N-1}})\}]
\end{array}
\quad \text{- (D)}
$$

From the set of linear equations **(D)** one can find $\begin{bmatrix} c_2 \\ c_2 \\ . \\ . \\ . \\ . \\ c_{N-1} \end{bmatrix}$ using matrix mathetics (see reference to tridiagonal system of matirices).

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Thus as a first step the c_i coefficient s are obtained from the a_i coefficient s and then the remaining coefficients are obtaied using equations (A) and (B).
Once all coefficients are obtained we have our zero-rate curve.

References:
1 "Fitting the term structure of interest rates: the practical implementation of cubic spline methodology"
http://www.yieldcurve.com/Mktresearch/files/PienaarChoudhry_CubicSpline2.pdf
2 "Spline methods for extracting interest rate curves from coupon bond prices"
http://www.frbatlanta.org/filelegacydocs/Wp9710.pdf
http://en.wikipedia.org/wiki/Par_yield
http://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm
http://en.wikipedia.org/wiki/Natural_cubic_spline